U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE
NATIONAL METEOROLOGICAL CENTER

OFFICE NOTE 102

On the Contributions to Analysis Errors from Errors in Forecasts and Observations

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AUGUST 1974

I. Introduction

In this paper, the theory of objective analysis errors is developed in a simplified form in order to demonstrate that

(1) It is generally advantageous to weight the forecast value in some way with observational data in the objective analysis.

(2) Observational data derived from satellite radiance measurements possess certain inherent disadvantages when used in the conventional way in an objective analysis scheme.

Optimum interpolation objective analysis methods have been discussed by Gandin (1963), Bengtsson and Gustavsson (1971, 1972), and Rutherford (1972). Gandin analyzes departures of observational values from climatological norms with no reference made to forecast values. All observations are assumed to be of the same type and to possess random errors. Bengtsson and Gustavsson, and also Rutherford, analyze deviations of observational values from forecast values, also with the assumption of uniform observational type and random errors.

The above authors do not consider analysis problems associated with heterogeneous data sources. The advent of quantitative satellite data has made it necessary to consider the special problems associated with such data. The likelihood that satellite observational errors are spatially correlated along an orbital pass, and the fact that operational VTPR observations as currently determined are highly correlated with forecast values, both tend to reduce the informational value of the satellite observations.

Alaka and Elvander (1972) have considered the problem of optimum interpolation using observations of mixed quality; however, the error correlations peculiar to satellite data are not considered in their treatment. The present study focuses upon these error correlations and the role which they play in determining the error of an optimized objective analysis. A simplified analysis "model" is introduced in order to obtain quantitative results without requiring a detailed knowledge of the structure of data fields. Although this simplification necessarily restricts the application of the results, it is nevertheless hoped that, at least in an approximate sense, the results correctly indicate the nature of the impact of satellite observations upon objective analysis error.

II. The Analysis Model

In objective analysis methods, errors arise for two different reasons. First, the interpolation method used, be it polynomial approximation, distance dependent weighting functions (as in the Cressman (1959) successive corrections scheme), or optimum interpolation in the manner of Gandin, leads to error since no interpolation method can exactly obtain the "true" value at an analysis grid point. The magnitude of this error depends, in general, upon the density and distribution of observations in the vicinity of the grid point as well as on the scale of disturbances analyzed. This error will generally differ from one grid point to another.

The second source of analysis error is the error of the observations themselves (and also of the forecast value if used in the analysis). This error of course differs from one observation to another, and the effect of observational errors on the resulting analysis error will also differ from grid point to grid point. However, individual observational errors are rarely known, therefore we must use statistical values of these errors. In this case, the error in the analysis at a particular grid point which is due to observational error may be determined, in a statistical sense, from the number and kind of observations used to obtain the analyzed value at the grid point. It is with this contribution to the total analysis error that the present study is concerned.

Assume that a currently valid forecast value z_f of a meteorological variable z is available at a given analysis grid point. Additionally, assume that there are m rawinsonde observations and n satellite observations in the vicinity of the grid point which are used to obtain, by interpolation, the analyzed value z_o at the grid point. A linear interpolation formula which evaluates the analyzed value in terms of the forecast value plus the (m + n) observations will be of the general form

$$z_o = \sum_{i=1}^{m} a_i (z_r)_i + \sum_{j=1}^{n} b_j (z_s)_j + c z_f , \tag{1}$$

where the a_i, b_j, and c are coefficients for assigning relative weights to the observations and the forecast. We will require that the weighting coefficients be non-negative and that

$$\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j + c = 1. \tag{2}$$

It may then be shown that the error variance of the analyzed value due to the errors of the observations and forecast is given in general by

$$\sigma_o^2 = \sigma_r^2 \sum_{i=1}^{m} \sum_{k=1}^{m} \rho_{ik} a_i a_k + \sigma_s^2 \sum_{j=1}^{n} \sum_{\ell=1}^{n} \rho_{j\ell} b_j b_\ell + c^2 \sigma_f^2$$

$$+ \sigma_r \sigma_s \sum_{i=1}^{m} \sum_{j=1}^{n} \rho_{ij} a_i b_j + 2c\sigma_f \left[\sigma_r \sum_{i=1}^{m} \rho_{fi} a_i + \sigma_s \sum_{j=1}^{n} \rho_{fj} b_j \right], \qquad (3)$$

where the σ's are the standard deviations of the errors, and the ρ's are correlation coefficients between the errors of pairs of observations, or between errors in the observations and the grid point forecast, as indicated.

It may be assumed that the rawinsonde observational errors are uncorrelated with each other, with the satellite errors, or with the forecast error. Then (3) reduces to

$$\sigma_o^2 = \sigma_r^2 \sum_{i=1}^{m} a_i^2 + \sigma_s^2 \left[\sum_{j=1}^{n} b_j^2 + \sum_{j \neq \ell} \sum \rho_{j\ell} b_j b_\ell \right] + c^2 \sigma_f^2$$

$$+ 2c\sigma_f \sigma_s \sum_{j=1}^{n} \rho_{fj} b_j . \qquad (4)$$

The retained correlation coefficients assume that the satellite observational errors are positively correlated with each other and may be individually correlated with the forecast error at the grid point.

Introducing the definitions

$$\rho_{ss} \equiv \frac{\sum\sum_{j \neq l} \rho_{jl} \, b_j \, b_l}{\sum\sum_{j \neq l} b_j \, b_l} \, , \tag{5}$$

$$\rho_{fs} \equiv \frac{\sum_{j=1}^{n} \rho_{fj} \, b_j}{\sum_{j=1}^{n} b_j} \tag{6}$$

in equation (6) gives

$$\sigma_o^2 = \sigma_r^2 \sum_{i=1}^{m} a_i^2 + \sigma_s^2 \left[\sum_{j=1}^{n} b_j^2 + \rho_{ss} \sum\sum_{j \neq l} b_j \, b_l \right]$$

$$+ c^2 \sigma_f^2 + 2c \, \sigma_f \, \sigma_s \, \rho_{fs} \sum_{j=1}^{n} b_j \, , \tag{7}$$

where ρ_{ss} and ρ_{fs} may be interpreted as equivalent weighted correlation coefficients. The autocorrelation ρ_{ss} is the weighted mean spatial correlation of satellite errors for those satellite observations used to determine the analyzed value at the grid point.

The correlation coefficient ρ_{fs} refers to the weighted mean correlation between each of the satellite observational errors and the forecast error at the grid point being analyzed. Note that this is not the same as the correlation between satellite error and forecast error at the satellite observational location. If ρ_{Fs} represents this latter correlation, then the two are related by

$$\rho_{fs} = \rho_{Fs} \cdot \rho_{Ff} \tag{8}$$

where ρ_{Ff} is the weighted mean correlation of the forecast error at the satellite observational location with that at the grid point location. The magnitude of ρ_{Ff} would likely be determined by the ratio of the area scanned for the grid point analysis to that of synoptic-scale disturbances. If the satellite observations are all relatively close to the grid point being analyzed, the value of ρ_{Ff} would be expected to be nearly unity. It is clear that ρ_{fs} vanishes when ρ_{Fs} is zero for all the satellite observations, no matter what value ρ_{Ff} has.

Finally, we define

$$\bar{a} \equiv \frac{1}{m} \sum_{i=1}^{m} a_i \,, \tag{9}$$

$$\bar{b} \equiv \frac{1}{n} \sum_{j=1}^{n} b_j \,, \tag{10}$$

and re-express (1), (2), and (7) in terms of $\overline{a}$ and $\overline{b}$. The resulting expressions are

$$z_o = \overline{a} \sum_{i=1}^{m} (z_r)_i + \overline{b} \sum_{j=1}^{n} (z_s)_j + c z_f + \sum_{i=1}^{m} (\overline{a} - a_i)(z_r)_i + \sum_{j=1}^{n} (\overline{b} - b_j)(z_s)_j \, , \tag{11}$$

$$m\overline{a} + n\overline{b} + c = 1 \, , \tag{12}$$

$$\sigma_o^2 = m\sigma_r^2 \left[\overline{a}^2 + var\,(a_i) \right] + n\sigma_s^2 \left\{ \overline{b}^2 + var\,(b_j) + (n-1)\rho_{ss} \left[\overline{b}^2 + cov\,(b_j, b_\ell) \right] \right\} + c^2\sigma_f^2 + 2n\overline{b}c\,\sigma_f \sigma_s \rho_{fs} \, . \tag{13}$$

In general, the values of var (a_i), var (b_j), and cov (b_j, b_ℓ) will vary from grid point to grid point as the values of the a and b coefficients themselves vary. It is interesting to note that σ_o^2 as given by (13) is a minumum (other quantities fixed) when

$$var\,(a_i) = var\,(b_j) = cov\,(b_j, b_\ell) = 0 \, . \tag{14}$$

This corresponds to <u>equal weighting</u> of all rawinsonde observations and also of all satellite observations such that

$$a_i = \bar{a}, \quad b_j = \bar{b} \tag{15}$$

for all i and j.

Although such a choice of weighting coefficients minimizes the contribution of the observational and forecast errors to the total analysis error, it does not necessarily result in the minimum value of the total analysis error. The contribution to the analysis error by the error in the interpolation will depend upon the choice of the individual coefficients a_i and b_j for the weighting of observations in the vicinity of a grid point. This latter contribution will not, in general, be a minimum when (15) is fulfilled.

Using optimum interpolation based on climatological statistics, Gandin (1963) has obtained general expressions for the minimized total analysis error and for the corresponding optimized values of the weighting coefficients. His results are applicable only for observations of uniform quality which possess completely random, uncorrelated errors. Quantitative values for the analysis error and for the corresponding weighting coefficients can be obtained only for a specified distribution of observations in the vicinity of a grid point. In an actual analysis situation, this distribution, along with the optimized weighting coefficients and the resulting analysis error, will of course change from one grid point to another.

In this study, we effectively eliminate the interpolative contribution to the analysis error by assuming that the meteorological field being analyzed is a uniform one. In this case, interpolation by (11) will be exact, except for the effect of observational and forecast errors, no matter how the weighting coefficients a_i, b_j, and c are chosen. We will choose the a_i and b_j coefficients according to (15) so that the variances and covariances of the coefficients in (13) will vanish. (An alternative analysis model producing the same mathematical result is colocation of all observations with the grid point. In this case, the autocorrelation ρ_{ss} would not be interpreted as a spatial correlation of satellite errors.)

Substituting (15) in (13) gives

$$\sigma_o^2 = m\bar{a}^2\sigma_r^2 + n\bar{b}^2\sigma_s^2\left[1+(n-1)\rho_{ss}\right] + c^2\sigma_f^2 + 2n\bar{b}c\,\sigma_f\sigma_s\rho_{fs}. \tag{16}$$

We now choose the coefficients $\bar{a}$, $\bar{b}$, and c, subject to the constraint of equation (12), such that σ_o^2 assumes its minimum value. Using the method of Lagrange multipliers,

$$\frac{\partial\sigma_o^2}{\partial(\bar{a},\bar{b},c)} + \lambda\frac{\partial\phi}{\partial(\bar{a},\bar{b},c)} = 0 \tag{17}$$

where $\phi \equiv m\bar{a} + n\bar{b} + c - 1 = 0$.

Application of (17) to (16) yields the set of equations:

$$2m\bar{a}\sigma_r^2 + m\lambda = 0 \tag{18a}$$

$$2nk\bar{b}\sigma_s^2 + 2nc\rho_{fs}\sigma_f\sigma_s + n\lambda = 0 \tag{18b}$$

$$2c\sigma_f^2 + 2n\bar{b}\rho_{fs}\sigma_f\sigma_s + \lambda = 0 \tag{18c}$$

where, for notational convenience,

$$k \equiv 1 + (n-1)\rho_{ss} \tag{19}$$

The parameter k is thus a measure of the degree of mutual correlation of the satellite observational errors.

Solution of the system of equations (18a, b, c) and (12) leads to the following expressions for the coefficients:

$$\bar{a} = \sigma_f^2\sigma_s^2(k - n\rho_{fs})/D \tag{20a}$$

$$\bar{b} = \sigma_r^2(\sigma_f^2 - \rho_{fs}\sigma_f\sigma_s)/D \tag{20b}$$

$$c = \sigma_r^2(k\sigma_s^2 - n\rho_{fs}\sigma_f\sigma_s)/D \tag{20c}$$

where

$$D \equiv n\sigma_r^2\sigma_f^2 + k\sigma_r^2\sigma_s^2 - 2n\rho_{fs}\sigma_r^2\sigma_f\sigma_s + m(k - n\rho_{fs}^2)\sigma_f^2\sigma_s^2 \tag{21}$$

Finally, substitution of (20a, b, c) in (16) yields, for the minimized value of σ_o^2,

$$\sigma_o^2 = \sigma_r^2 \sigma_s^2 \sigma_f^2 \left(k - n\rho_{fs}^2\right) / D . \tag{22}$$

It should be noted that the denominator D is positive-definite.

For graphical and computational purposes, it is convenient to re-express (20a, b, c) and (22) in a normalized form in terms of the error ratios

$$\alpha \equiv \sigma_r / \sigma_f , \qquad \beta \equiv \sigma_s / \sigma_f . \tag{23}$$

With these definitions, equations (20a, b, c) and (22) may be expressed as

$$\bar{a} = \left(k - n\rho_{fs}^2\right) \left(\beta/\alpha\right)^2 / \Delta \tag{24a}$$

$$\bar{b} = \left(1 - \rho_{fs}\beta\right) / \Delta \tag{24b}$$

$$c = \left(k\beta^2 - n\rho_{fs}\beta\right) / \Delta \tag{24c}$$

$$(\sigma_o/\sigma_f)^2 = \beta^2(k - n\rho_{fs}^2)/\Delta \tag{25}$$

where

$$\Delta \equiv n + k\beta^2 - 2n\rho_{fs}\beta + m(k - n\rho_{fs}^2)(\beta/\alpha)^2.$$

The solutions given by (20a, b, c) and (22), or alternatively by (24a, b, c) and (25), do not apply for all possible values of β and positive ρ_{fs}. Equation (24b) gives a negative value of $\overline{b}$ when $\beta > \frac{1}{\rho_{fs}}$, whereas equation (24c) gives a negative value of c when $\beta < \frac{n}{k}\rho_{fs}$. As only non-negative weights are permitted (a negative weight assigned to real data would be a questionable practice), we conclude that, when $\rho_{fs} > 0$, equations (24a, b, c) and (25) are valid only when

$$\frac{n}{k}\rho_{fs} \leq \beta \leq \frac{1}{\rho_{fs}}. \tag{26}$$

However, $\frac{n}{k}\rho_{fs} \leq \frac{1}{\rho_{fs}}$ only if $\rho_{fs}^2 \leq \frac{k}{n}$. Thus, equations (24a, b, c) and (25) are valid only provided this additional criterion is met:

$$0 \leq \rho_{fs} < \sqrt{k/n}. \tag{27}$$

These restrictions do not apply when ρ_{fs} is negative; however, we are only concerned with the implications of positive correlation in this study.

The question arises as to how the results should be modified when β and/or ρ_{fs} are outside of the ranges specified by (26) and (27). The answer is that either the forecast value z_f or the satellite observations must be discarded in order to obtain minimum analysis error. For condition (27) satisfied but $\beta < \frac{n}{k}\rho_{fs}$, we set c=0 in equations (12) and (16) and minimize

$$\sigma_o^2 = m\bar{a}^2\sigma_r^2 + nk\bar{b}^2\sigma_s^2 \tag{28}$$

with respect to the coefficients $\bar{a}$ and $\bar{b}$, subject to the condition that

$$m\bar{a} + n\bar{b} = 1 \tag{29}$$

The resulting expressions are

$$\bar{a} = \frac{k\sigma_s^2}{n\sigma_r^2 + mk\sigma_s^2} = \frac{k(\beta/\alpha)^2}{n + mk(\beta/\alpha)^2} \tag{30a}$$

$$\bar{b} = \frac{\sigma_r^2}{n\sigma_r^2 + mk\sigma_s^2} = \frac{1}{n + mk(\beta/\alpha)^2} \tag{30b}$$

$$\sigma_o^2 = \frac{k\sigma_r^2\sigma_s^2}{n\sigma_r^2 + mk\sigma_s^2}\,; \quad (\sigma_o/\sigma_f)^2 = \frac{k\beta^2}{n + mk(\beta/\alpha)^2} \tag{31}$$

For condition (27) satisfied but $\beta > \frac{1}{\rho_{fs}}$, we set $\bar{b}=0$ in equations (12) and (16) and proceed as above. The resulting expressions in this case are

$$\bar{a} = \frac{\sigma_f^2}{\sigma_r^2 + m\sigma_f^2} = \frac{1}{\alpha^2 + m} \tag{32a}$$

$$c = \frac{\sigma_r^2}{\sigma_r^2 + m\sigma_f^2} = \frac{\alpha^2}{\alpha^2 + m} \tag{32b}$$

$$\sigma_o^2 = \frac{\sigma_r^2 \sigma_f^2}{\sigma_r^2 + m\sigma_f^2} \; ; \quad (\sigma_o/\sigma_f)^2 = \frac{\alpha^2}{\alpha^2 + m} . \tag{33}$$

The same results obtain when n=0 in equations (24a), (24c), and (25). Equation (33) also corresponds to the expression given by Gandin (1963, p.85) for the minimum possible interpolation error for homogeneous observations with random errors, except that (33) is normalized with respect to σ_f^2 rather than climatological variance. Note that $\sigma_o \approx \sigma_r / \sqrt{m}$ when the number m of observations becomes large.

When condition (27) is not satisfied, i.e., $\rho_{fs} \geq \sqrt{k/n}$, equation (25) does not give a real value for σ_o, hence a three-coefficient solution is not possible for any value of β. Also, a solution with $\bar{a}=0$ is not possible.

Hence the choice is between (20) - (31) with c=0 and (32) - (33) with b=0. By comparing the right-hand sides of equations (31) and (33), it is easy to show that (31) gives the minimum σ_o^2 when $\beta < \sqrt{n/k}$ and that (33) gives the minimum σ_o^2 when $\beta > \sqrt{n/k}$. The two solutions correspond when $\beta = \sqrt{n/k}$.

The various possibilities are shown diagrammatically in Figure 1. Here ρ_{fs} is plotted as the abscissa and β as the ordinate. The domain is divided into three regions by the curves $\beta = n\rho_{fs}/k$, $\beta = 1/\rho_{fs}$, and (for $\rho_{fs} \geq \sqrt{k/n}$) $\beta = \sqrt{n/k}$. Equations (24a, b, c) and (25) give minimum analysis error in region I, equations (30a, b) and (31) in region II, and equations (32a, b) and (33) in region III. For the special case of n=0 (no satellite observations), equations (32a, b) and (33) apply everywhere.

The figure indicates that, when the correlation ρ_{fs} between observational and forecast errors is small, the lowest analysis error is usually obtained by a three-way blend of forecast with rawinsonde and satellite observations. On the other hand, this is no longer true when ρ_{fs} is relatively large. Then, either the forecast value or the correlated observation must be discarded as being essentially redundant information with an error level which only results in degrading the analysis if used. The implication of this result for satellite VTPR data, believed to be highly correlated with forecast values, in analysis procedures is obvious.

When $\rho_{fs} = 0$, equation (25) reduces to

$$(\sigma_o/\sigma_f)^2 = \frac{k\beta^2}{n + k\beta^2 + mk(\beta/\alpha)^2} \tag{32}$$

By comparing the right-hand side of this expression with the right-hand sides of (25), (31), and (33), it is easy to show that, for given σ_r, σ_s, and σ_f, the uncorrelated case ($\rho_{fs} = 0$) gives the lowest possible error.

The effect of the mutual or spatial correlation ρ_{ss} of satellite observational errors on the results enters through the parameter k. Recalling that

$$k \equiv 1 + (n-1)\rho_{ss} ,$$

we can see that k is unity if there is no spatial correlation (or only one satellite observation) but otherwise increases both as the number of satellite observations and their degree of spatial error correlation increases. When n is much larger than unity, the ratio

$$k/n \simeq \rho_{ss} .$$

It is apparent from (25) and (31) that, other factors being equal, increased spatial correlation of observational errors results in larger analysis error. It must be remembered that point values of the satellite observations are assumed in reaching this result. If <u>differences</u> between adjacent observational values are used instead in performing the analysis, the spatial error-correlation may be used to advantage. For if

$$\Delta z = (z_s)_2 - (z_s)_1 , \tag{35}$$

where $(z_s)_1$ and $(z_s)_2$ are two adjacent satellite observations, then it may be shown (by application of equation (3) for two satellite observations only) that the error in Δz is given by

$$\sigma_{\Delta z}^2 = 2\sigma_s^2 (1 - \rho_{12}) . \tag{36}$$

This result indicates that it may be advantageous to use differences rather than absolute values in analysis work provided the error correlation between pairs of observations is greater than 0.5. Thus gradient quantities, such as geostrophic or thermal winds, may be analyzed in preference to height or temperature fields.

III. Discussion of Results

Figures 2 through 8 show the dependence of the normalized analysis error σ_0/σ_f on the normalized satellite observational error β and on the mean correlation coefficient ρ_{fs} between the satellite observational errors and the forecast error at the grid point. These figures are divided by the solid lines into three regions as indicated schematically in Figure 1. In region I, the forecast is optimally weighted with both types of observations, in region II the forecast is discarded, and in region III the satellite observations are discarded. For the rawinsonde observations, a value of 0.5 has been assumed throughout for the normalized error α. In other words, the rawinsonde observations are assumed to have error levels which are half those of the forecast errors. Figures 2 through 4 show minimized analysis errors when m=0, i.e., there is no rawinsonde observation within the scan area about the analysis grid point in question. Figures 5 through 7 give minimized analysis errors for one rawinsonde observation within the scan area. Within these two sets of figures, the number n of satellite observations is 1, 5, or 25. Figure 8 shows minimized analysis errors when the number of both rawinsonde and satellite observations is 5.

Although it is believed that satellite observational errors are highly correlated with each other along a given orbital pass, the actual magnitudes of these correlations have not been measured. A value of 0.5 has been assumed for the mean spatial error correlation ρ_{ss} in Figures 3 through 8. (The satellite observations are assumed to be absolute, rather than difference, values.) Figures 9 and 10 are the same as Figures 3 and 4 except that ρ_{ss} is assumed to be zero. Figures 11 and 12 are the same as Figure 3 except that $\rho_{ss} = 0.25$ in Figure 11 and $\rho_{ss} = 0.75$ in Figure 12.

All of the figures show that it is clearly advantageous to weight the forecast value with the observations for those combinations of β and ρ_{fs} which lie in regions I or III. For example, suppose we have five satellite observations whose errors are uncorrelated with those of the forecast, along with one rawinsonde observation. Assuming that the satellite observations have errors equal in magnitude to those of the forecast ($\beta = 1$), Figure 6 gives a normalized analysis error of .387 if the forecast is optimally weighted with the observations, whereas the analysis error would be .420 if only the observations are optimally weighted.

Furthermore, if we have only a rawinsonde observation and the forecast to consider, region III of Figure 6 indicates that σ_o/σ_f is .447 when the rawinsonde is optimally weighted with the forecast. This is less than the assumed value of 0.5 for σ_r/σ_f. Similarly, five rawinsonde observations weighted with the forecast gives an analysis error of .218 (Figure 8) as compared with the error of .224 which results from using the observations only.

On the other hand, it would be disadvantageous to weight the forecast with the observation when (β, ρ_{fs}) lies in region II. Returning to the original example of five satellite observations and one rawinsonde observation, with $\beta = 1$ but with $\rho_{fs} > 0.6$, then inclusion of the forecast in any kind of weighting scheme would result in an analysis error greater than .420.

From the above discussion, it is apparent that the actual amount of improvement, if any, obtained in the analysis by including the forecast in the weighting scheme depends on the relative magnitudes of observational and forecast errors, and on the degree of correlation between them, as well as upon the number of observations used.

The disadvantages of satellite observational data in an analysis scheme are of three kinds:

(1) Current levels of satellite observational error are large compared to rawinsonde and other conventional error levels, and are comparable in magnitude to forecast errors.

(2) Satellite errors are believed to be spatially correlated with one another along an orbital pass. This lessens the informational input of satellite observations when analyzed in the conventional way.

(3) At least for the VTPR satellite data, errors are highly correlated with forecast errors. Hence the satellite observations tend to be redundant if forecast values are used in the analysis scheme.

It is apparent from all of the figures that the analysis error increases as the correlation ρ_{fs} increases until either the forecast or the satellite observation must be discarded. This result is of some significance since the current VTPR satellite data retrieval method produces data whose errors are highly correlated with those of the forecast. In terms of the quantities of equation (8), the correlation ρ_{Fs} between the satellite error and the forecast error at the same point is variously estimated as being in the range 0.5 to 0.8. The corresponding values of ρ_{fs} will be somewhat less than this; how much less depending on the mean value of ρ_{Ff} for the locations of the satellite observations used in the analysis at the grid point. VTPR satellite observations are spaced approximately 600 km apart in the oceanic areas; this is considerably less than the dimensions of synoptic disturbances, therefore ρ_{Ff} is probably not much less than unity for VTPR data.

Clearly, it is advantageous to use satellite data which is retrieved independently of a forecast "first guess" if the error levels are otherwise the same. In Figure 2, where one satellite observation is weighted with the grid point forecast, the analysis error is .707 for uncorrelated errors as compared to .866 for $\rho_{fs} = 0.5$ or .949 for $\rho_{fs} = 0.8$ (assuming that $\beta = 1$).

For those combinations of β and ρ_{fs} which lie in region III, the minimum analysis error is obtained by discarding the satellite observations. Inclusion of these observations with any positive weight in the analysis scheme would give larger analysis errors than those indicated in the figures. Since the errors in satellite temperature measurements have been estimated to be appreciably greater than those of the forecast errors in some cases (for example, over land areas where the forecast error is relatively small), and the forecast-observation error correlation has been estimated as high as 0.8 for VTPR data, it appears that the use of VTPR data in an uninformed way in an analysis scheme may actually result in degrading the resulting analysis. Of course if the ratio of satellite to forecast error β is small enough to lie in region II, then it is the forecast instead which should be discarded as being the redundant information with the higher error level.

Comparison of Figures 9, 11, and 12 with Figure 3 indicates the loss of analysis information that results when satellite observational errors are spatially correlated and the observations are treated as absolute point values. For example, given $\beta = 1.0$, $\rho_{fs} = 0.5$, Figure 9 indicates a value of .447 for σ_o/σ_f when $\rho_{ss} = 0$, Figure 11 gives .633 when $\rho_{ss} = .25$, Figure 3 gives .764 when $\rho_{ss} = .50$, and Figure 12 gives .829 when $\rho_{ss} = .75$. Similar trends are noted for other combinations of β and ρ_{fs}. (The actual value of ρ_{ss} which is applicable to current satellite data has yet to be determined.)

Intracomparison of Figures 2 through 4, 5 through 7, 6 with 8, and 9 with 10 illustrates the reduction of analysis error that results from increasing the number of observations used. In an actual analysis situation, this corresponds to increasing the density of available observations. The reduction of analysis error is most marked when the observational errors are not spatially correlated; compare Figures 9 and 10 with Figures 3 and 4. In the case of no spatial correlation, the analysis error is approximately proportional to $n^{-\frac{1}{2}}$. If the assumption that $\rho_{ss} = 0.5$ for the mean spatial correlation of satellite errors is approximately correct, then Figure 2 compared with Figure 4 and Figure 5 compared with Figure 7 indicate that reducing the satellite observational error by a factor of 2 would be more profitable than increasing the density of the satellite observations by a factor of 25!

In order to emphasize the importance of weighting observations with care in analysis work, it is worth comparing the analysis errors that would result from a "naive" weighting scheme such as giving equal weight to all observations used, regardless of type, and with the forecast discarded. In this case, equation (14) becomes

$$\sigma_o^2 = \frac{m\sigma_r^2 + nk\sigma_s^2}{(m+n)^2} \tag{35}$$

or, normalizing as before with respect to σ_f:

$$(\sigma_o/\sigma_f)^2 = \frac{m\alpha^2 + nk\beta^2}{(m+n)^2} \tag{36}$$

The square-root of this equation is plotted as a function of β in Figure 11 for m=0 and m=1 (with $\alpha = 0.5$), and for n=1, 5, and 25.

The upper part of the figure (for satellite observations only) always gives analysis errors equal to or greater than the corresponding errors of Figures 2, 3, and 4. Also, if the accuracy of the observations is poor, the analysis error will exceed the forecast error even if the density of such observations is very high. In fact, our assumed spatial correlation ρ_{ss} of 0.5 would make it impossible to reduce the analysis error below the forecast error, no matter how many observations are used, if the ratio β of satellite to forecast error is greater than two.

More interestingly, equal weighting of satellite observations and one conventional observation, as shown in the lower part of the figure, indicates that the resulting analysis would actually be worsened by the inclusion of additional satellite observations if β is greater than about 0.5. This would be a case of a relatively good observation being "swamped" by giving too much weight to satellite observations of relatively poorer quality.

Finally, it should be emphasized that the error level inherent in satellite data can probably be effectively reduced through the use of differences rather than absolute point values. Equation (34) indicates that the difference error $\sigma_{\Delta z}$ will be less than the individual observational error σ_s when the error correlation ρ_{12} between pairs of observations is greater than 0.5. For satellite data, this is most likely to be true when the two observations are close together and form part of the same orbital pass. It should be noted that the $\sigma_{\Delta z}$ difference errors themselves are likely to be spatially correlated along a given satellite orbital pass. Thus results similar to those of Figures 2 through 8 would still apply, except that the effective error level of the satellite data would be reduced.

IV. Concluding Remarks

An analytic treatment of analysis error has been developed which may be applied to the use of satellite data in objective analysis. In particular, both the spatial error correlation of satellite data and the correlation between satellite and forecast errors have been included in the derivations. Previous studies of analysis error by Gandin and others have usually assumed random, uncorrelated errors associated with observations of uniform type.

The contribution to total analysis error resulting from inexact interpolation has been neglected in the present treatment. A simplified analysis "model" has been assumed which gives, in effect, the minimum possible analysis error for any meteorological field once the observations have been optimally weighted. The observational data are assumed to be of two kinds, rawinsonde and satellite, and the forecast is also assumed to be given a relative weight in the analysis. The uniform field analysis model which is used necessarily restricts the generality of the results; nevertheless the author believes that the effects of observational and forecast errors on the resulting analysis error can best be indicated by stripping away the error resulting from inexact interpolation through use of this model.

The results of this study may be summarized as follows:

(1) Weighting a forecast value as an additional "observation" in the analysis scheme will frequently, but not always, reduce the resulting analysis error.

(2) When observational error levels are comparable, satellite data which are independent of the forecast result in lower analysis errors than do data retrieved in such a way that they are highly correlated with forecast values.

(3) Observations whose errors are correlated with but larger than forecast errors may, if care is not exercised in weighting the observations, inadvertently degrade the resulting analysis.

(4) The spatial correlation of errors that presumably is characteristic of satellite data is a handicap in conventional objective analysis schemes. The informational input of the observations is reduced as their spatial correlation increases. The "trade-off" which exists between the density of observations and their level of error is severely limited when the spatial correlation is high.

(5) Some of the problems associated with spatial correlation of satellite observational errors may be averted, or even used to advantage, if an analysis method that uses differences between adjacent satellite observations, such as gradient analysis, is employed.

ACKNOWLEDGEMENTS

The author wishes to acknowledge his debt to Dr. William Bonner for suggesting the original idea on which this study is based, and also his assistance in carrying out the study and in reviewing the manuscript.

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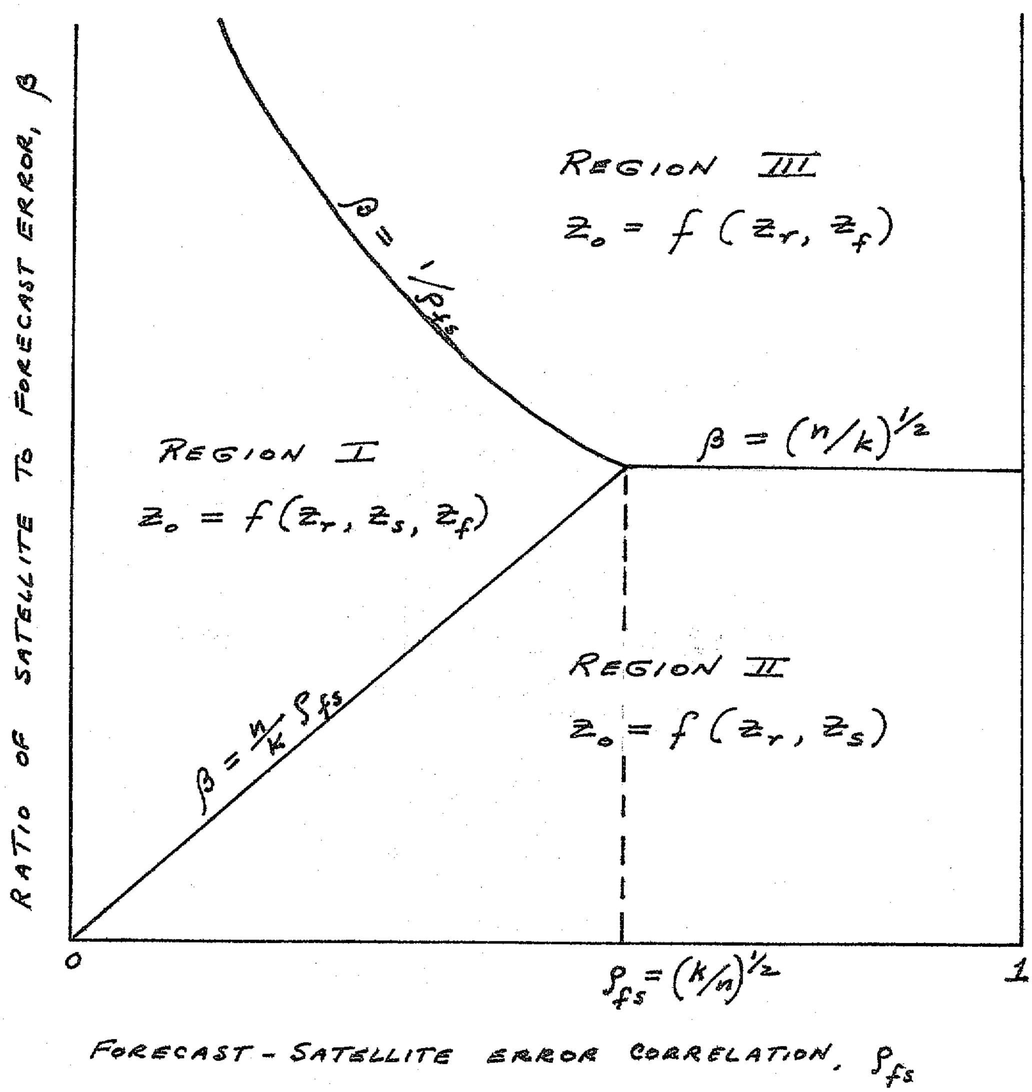

Fig. 1. Schematic plot of the ratio of satellite-to-forecast error, β, as a function of the forecast-satellite error correlation, ρ_{fs}, showing division of (β, ρ_{fs}) domain into three regions (see text).

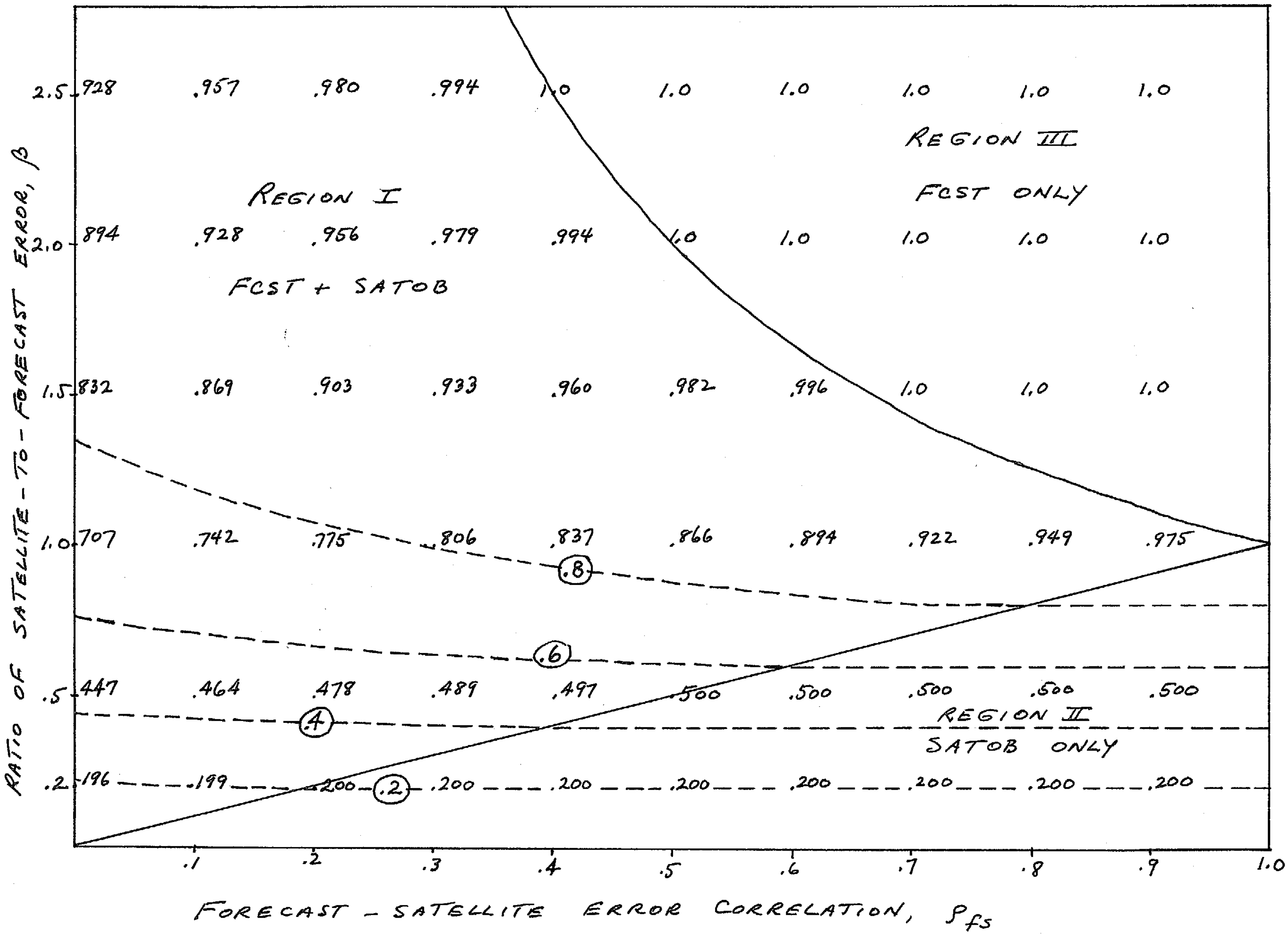

Fig. 2. Ratio of analysis error to forecast error, σ_o/σ_f, as a function of β and ρ_{fs} for m = 0; n = 1, k = 1.

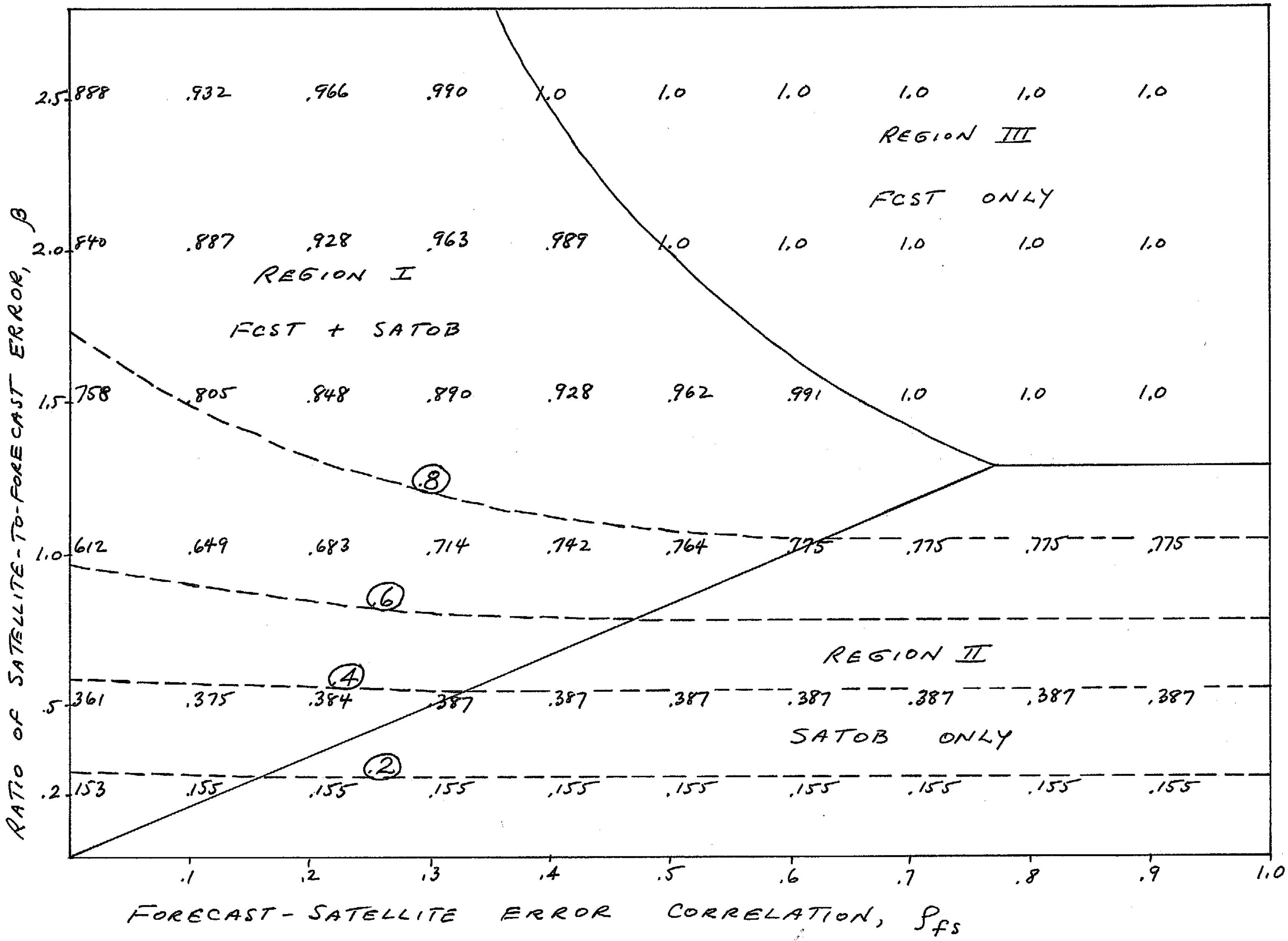

Fig. 3. Same as Fig. 2 but for m = 0; n = 5, ρ_{ss} = 0.5, k = 3.

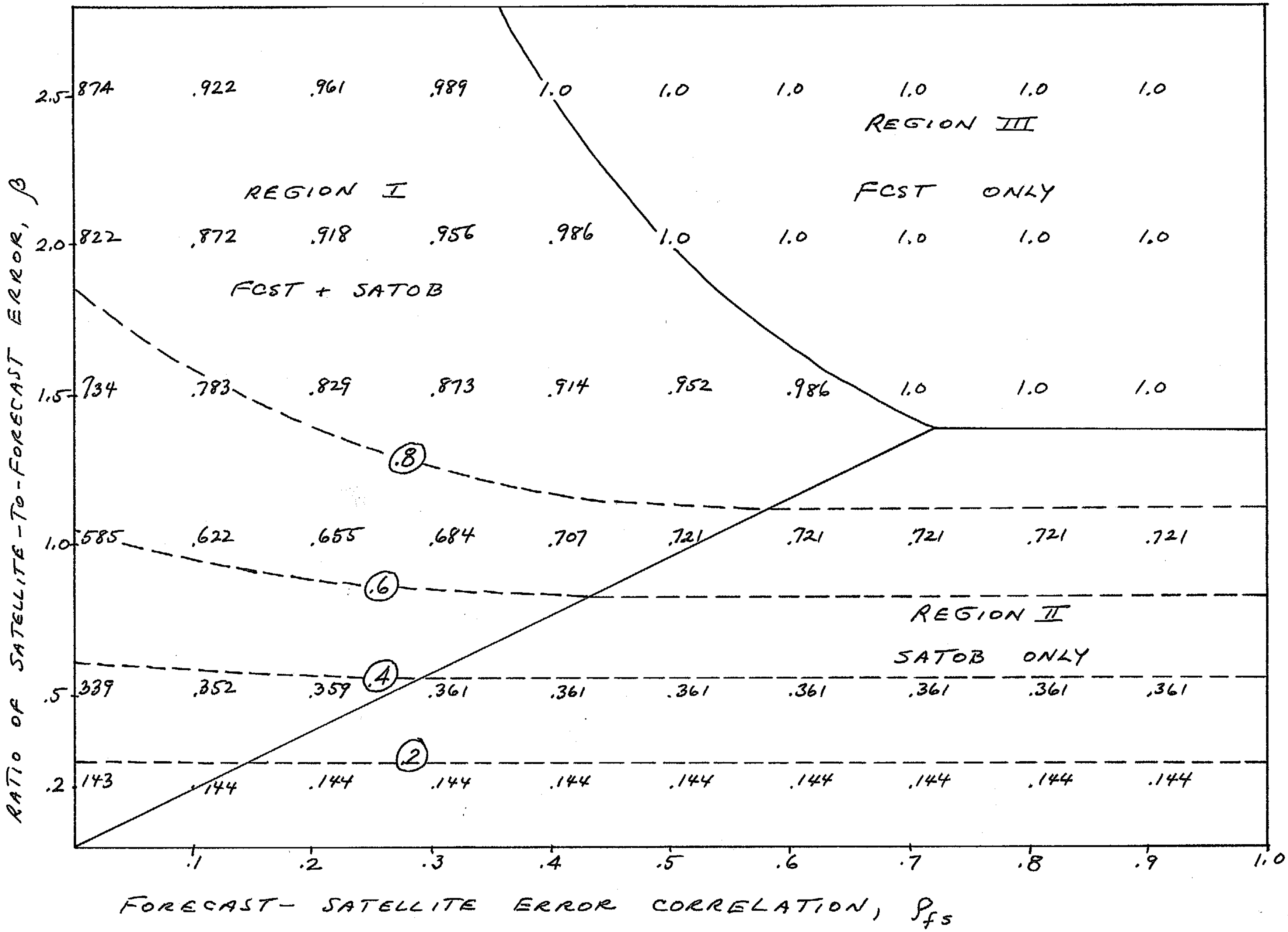

Fig. 4. Same as Fig. 2 but for m = 0; n = 25, ρ_{ss} = 0.5, k = 13.

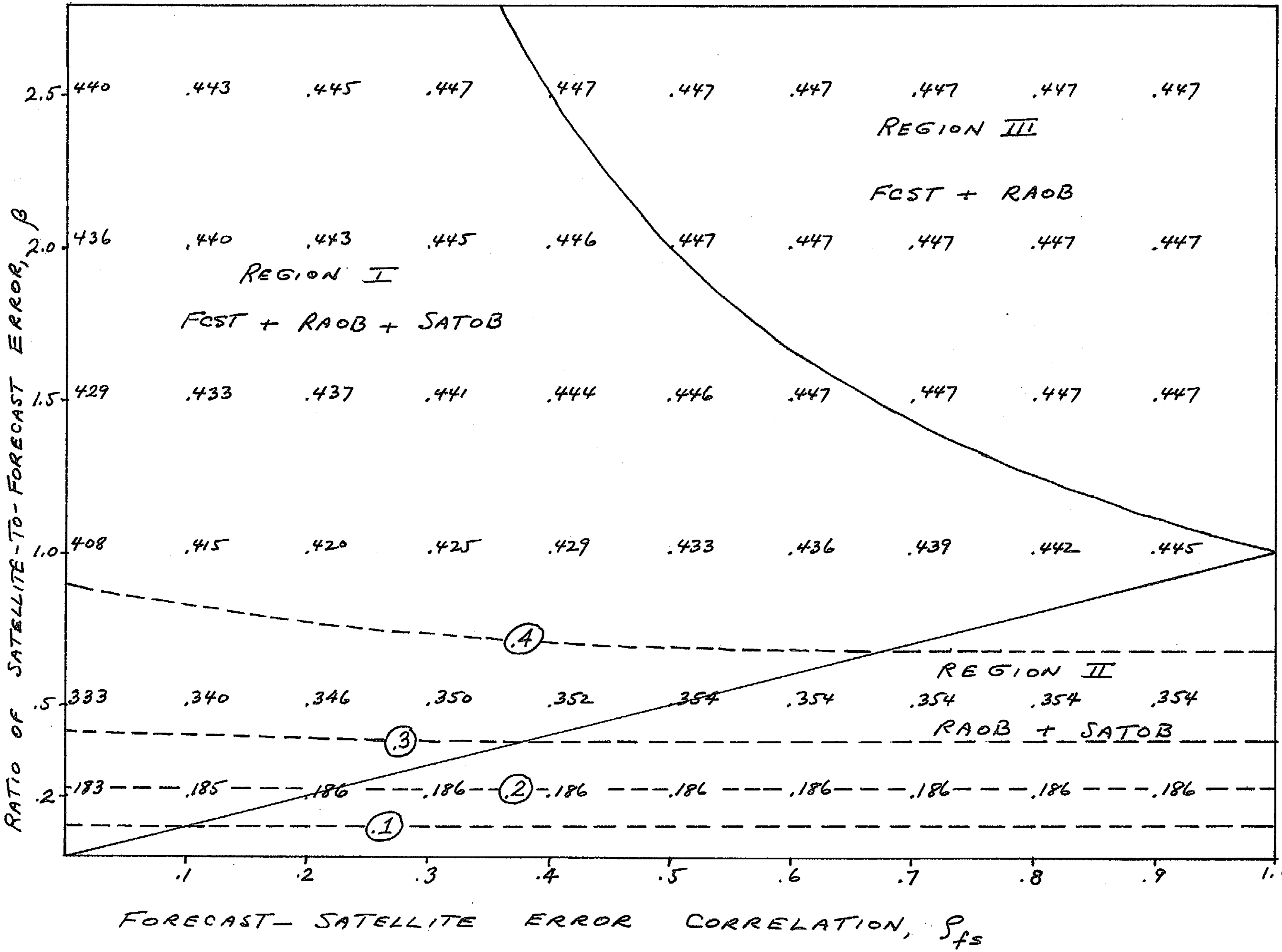

Fig. 5. Same as Fig. 2 but for m = 1, α = 0.5; n = 1, k = 1.

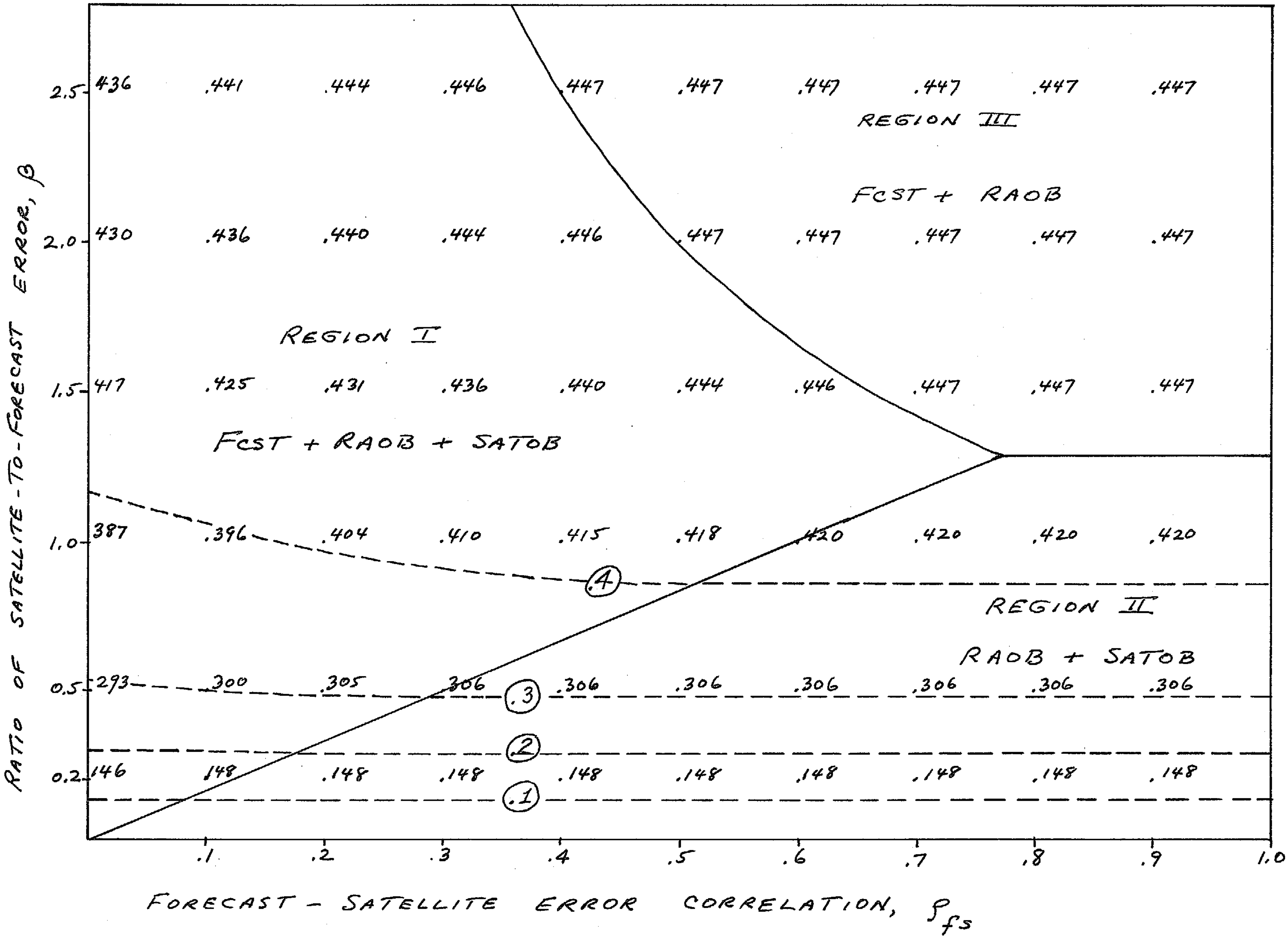

Fig. 6. Same as Fig. 2 but for m = 1, $\alpha = 0.5$; n = 5, $\rho_{ss} = 0.5$, k = 3.

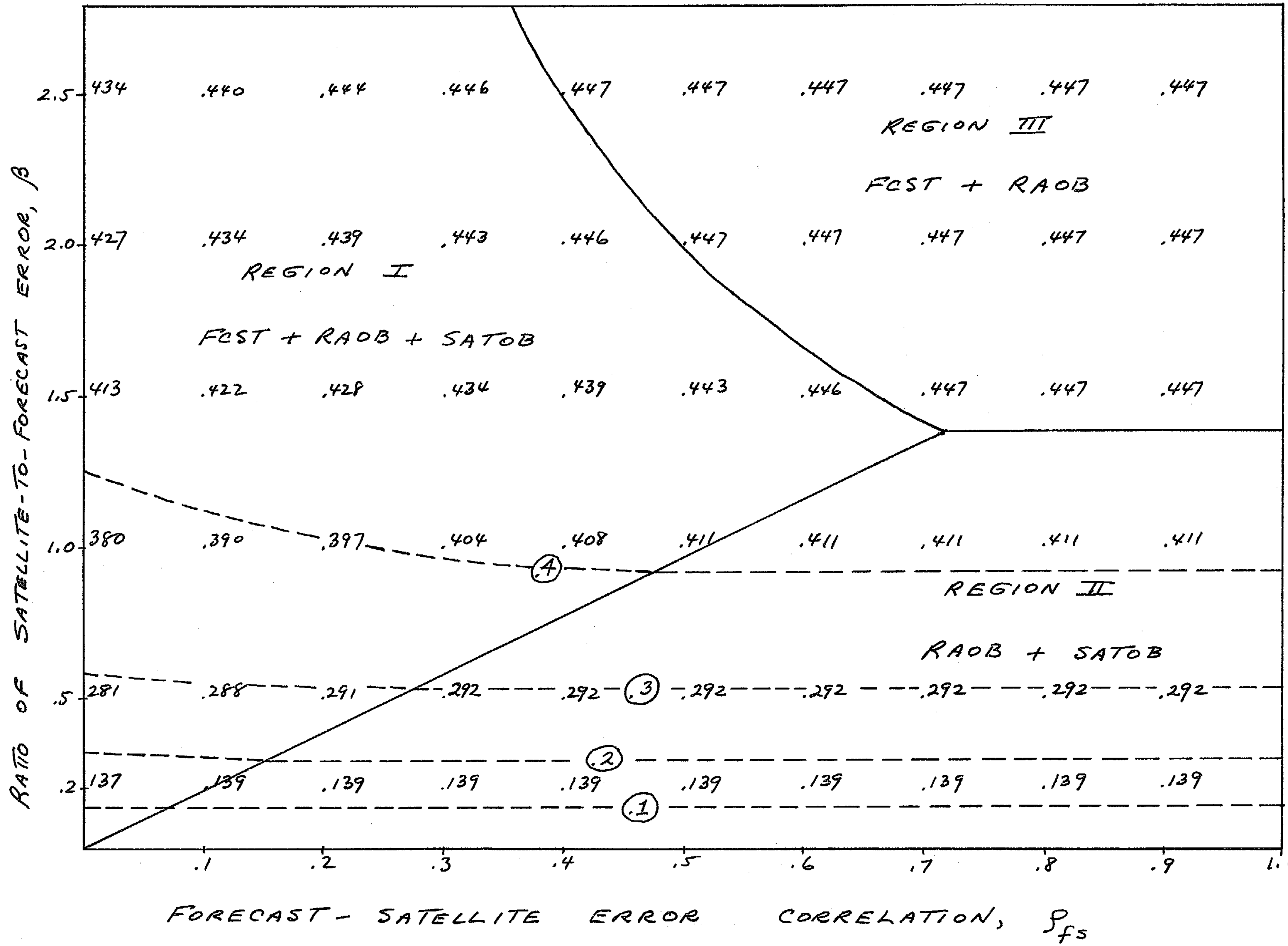

Fig. 7. Same as Fig. 2 but for m = 1, $\alpha = 0.5$; n = 25, $\rho_{ss} = 0.5$, k = 13.

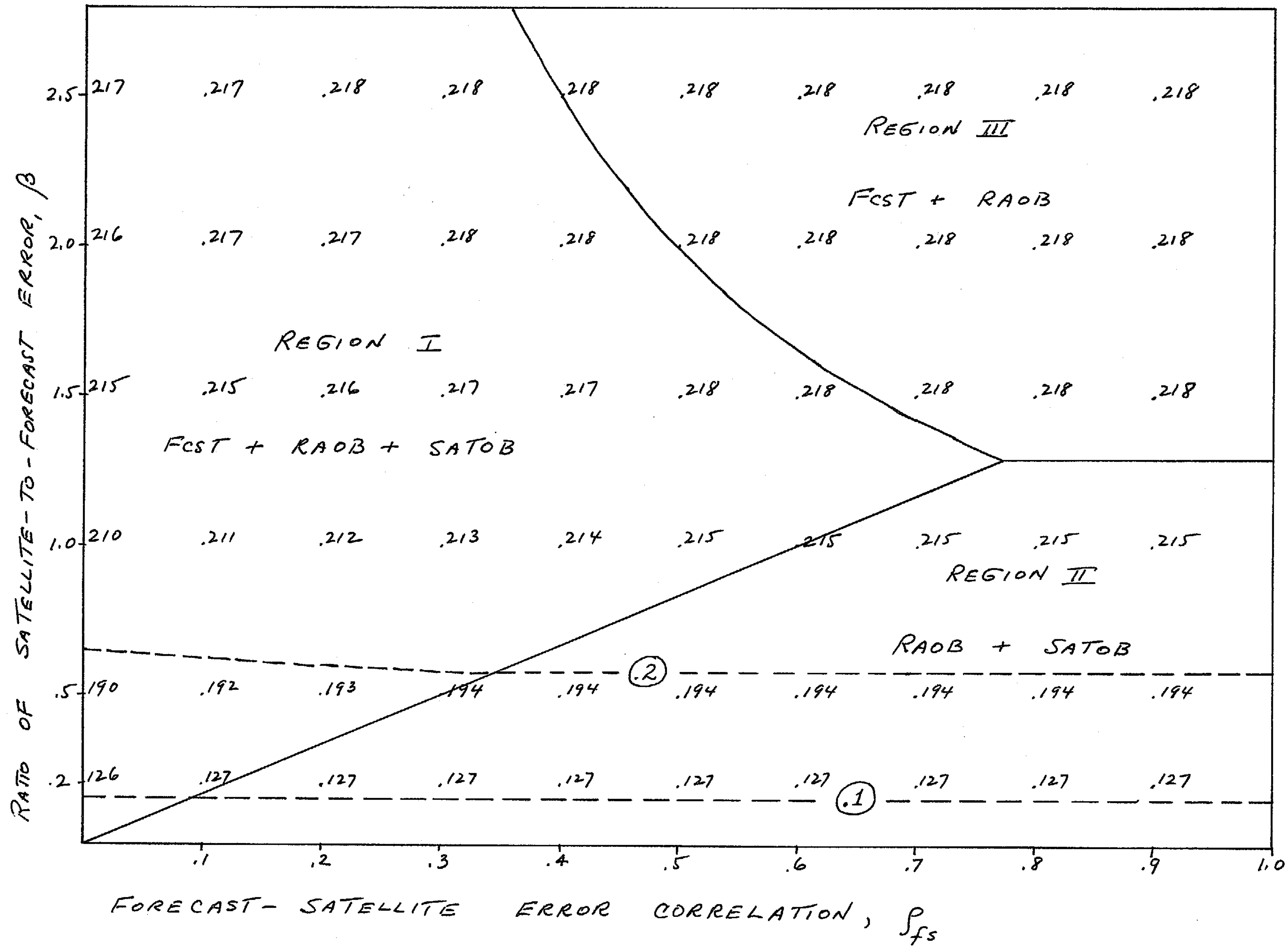

Fig. 8. Same as Fig. 2 but for m = 5, $\alpha = 0.5$; n = 5, $\rho_{ss} = 0.5$, k = 3.

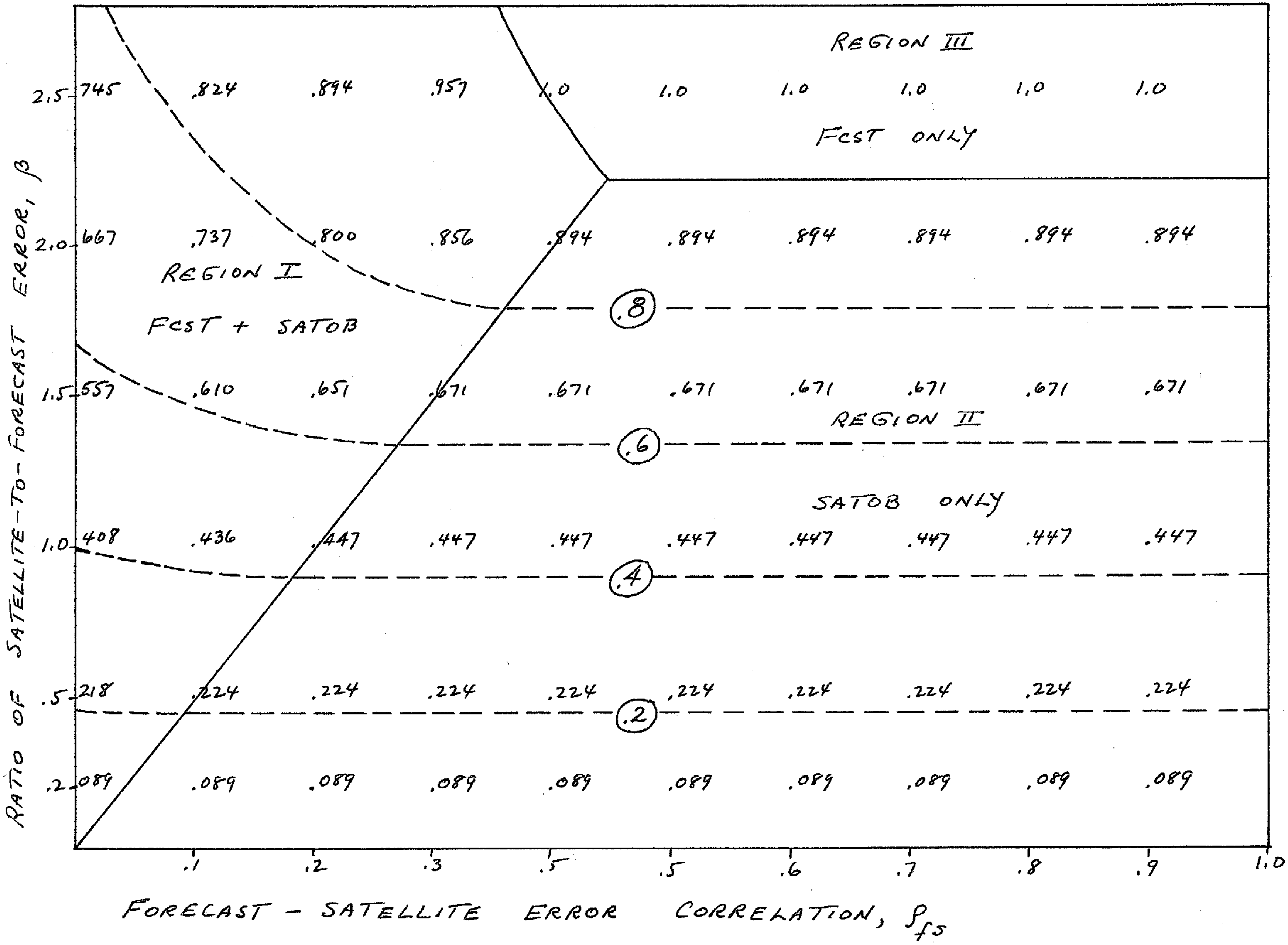

Fig. 9. Same as Fig. 2 but for $m = 0$; $n = 5$, $\rho_{ss} = 0$, $k = 1$.

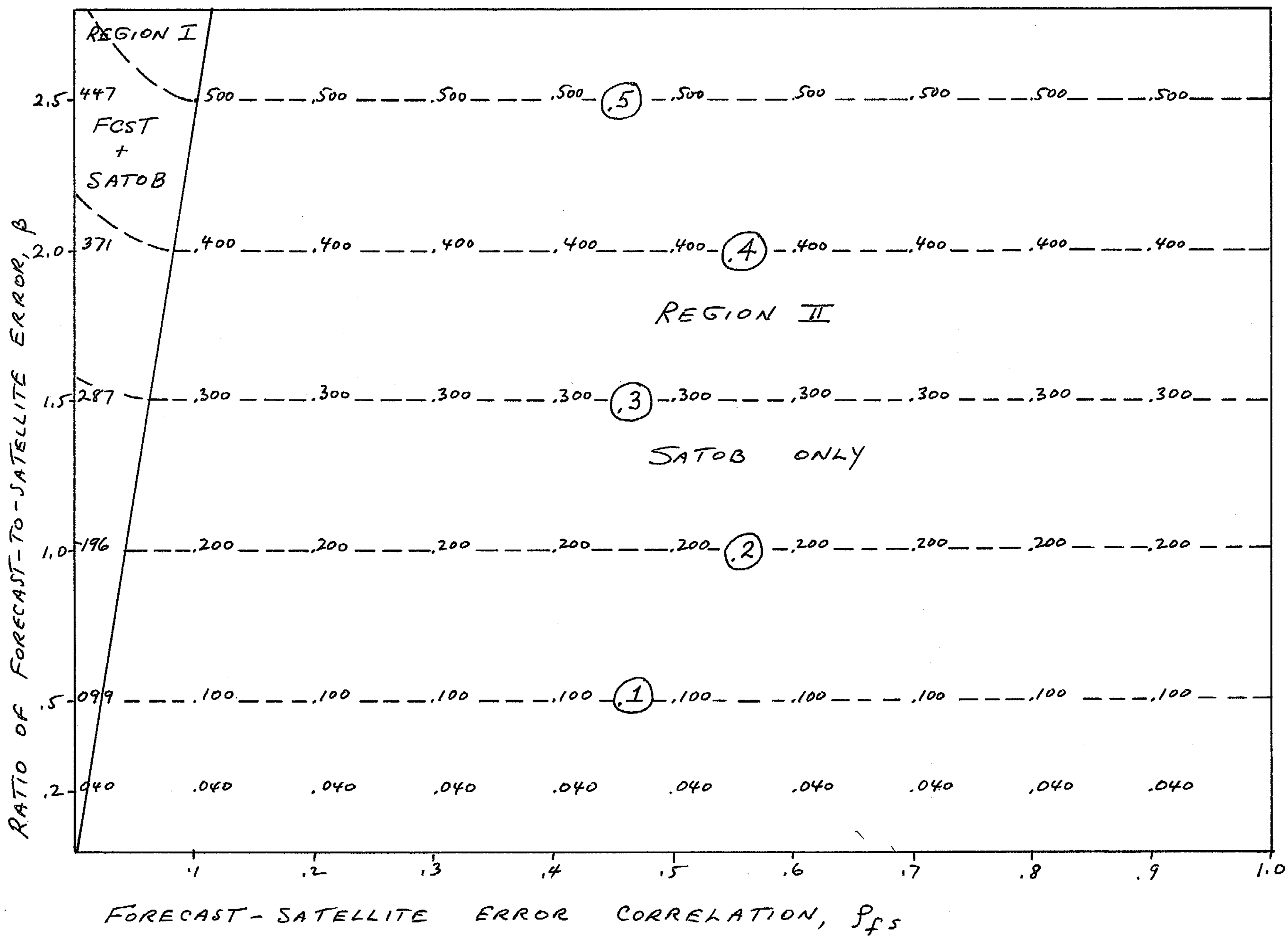

Fig. 10. Same as Fig. 2 but for m = 0; n = 25, ρ_{ss} = 0, k = 1.

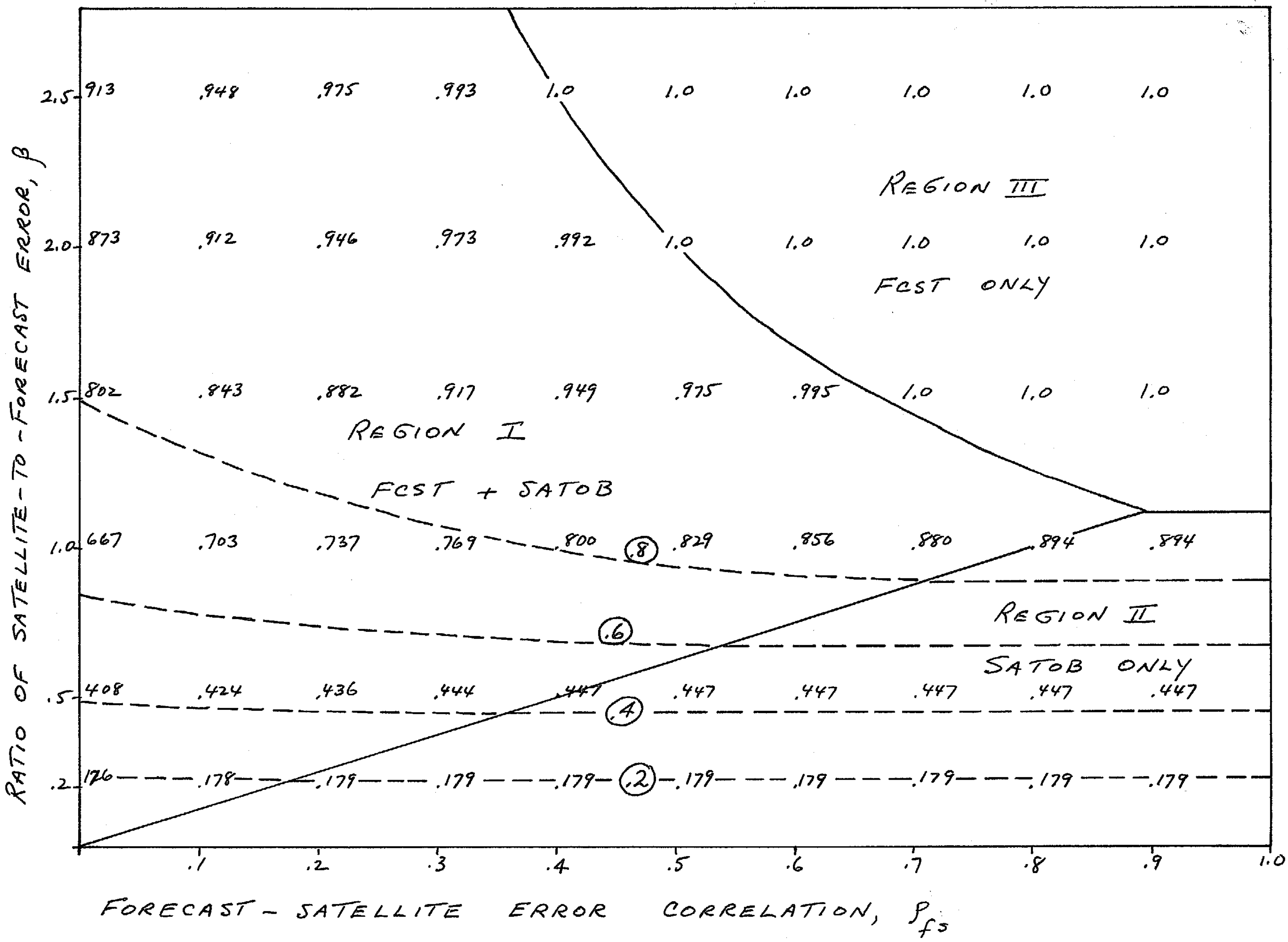

Fig. 12. Same as Fig. 2 but for m = 0; n = 5, ρ_{ss} = .75, k = 4.

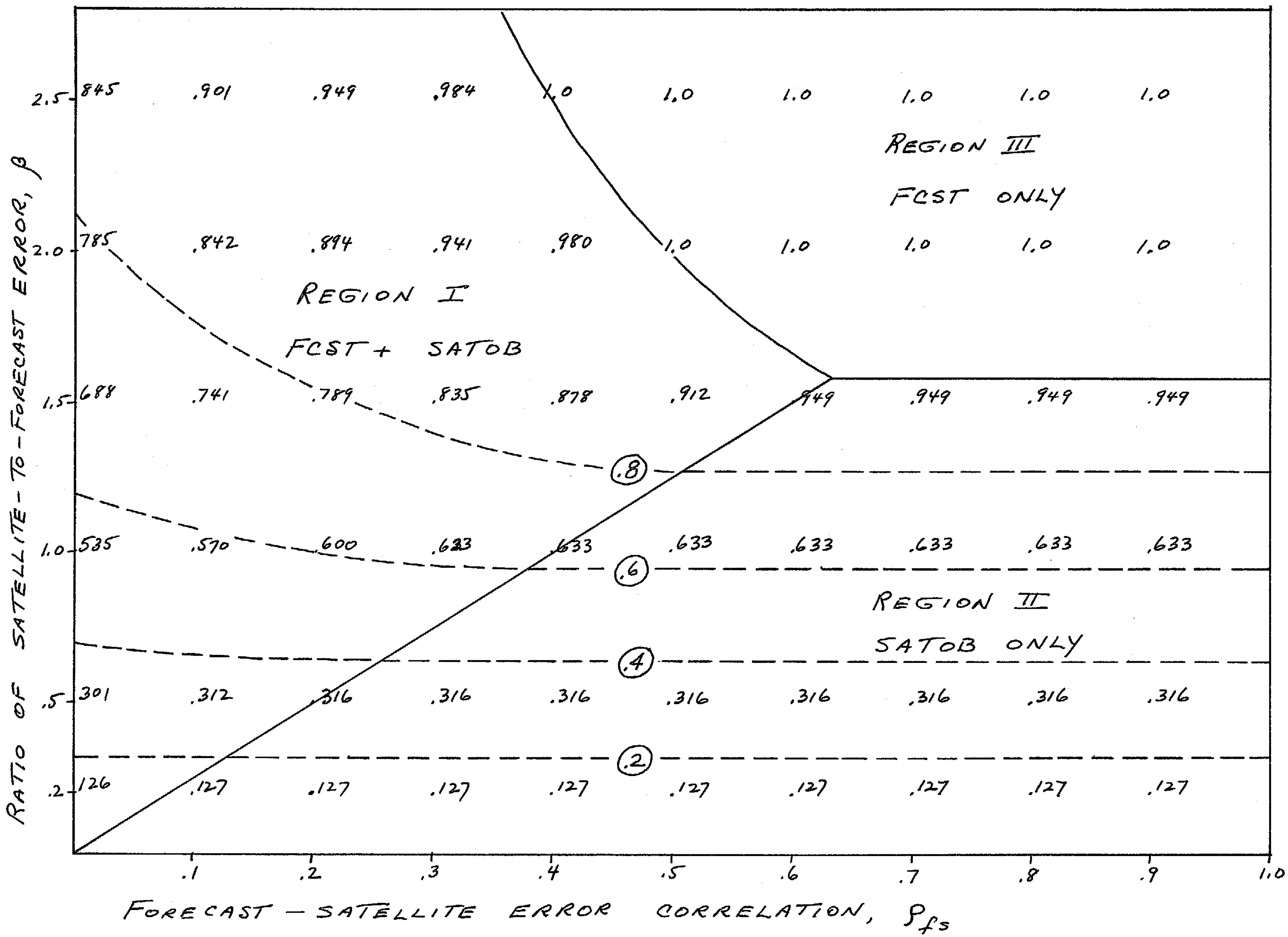

Fig. 11. Same as Fig. 2 but for m = 0; n = 5, ρ_{ss} = .25, k = 1.25.

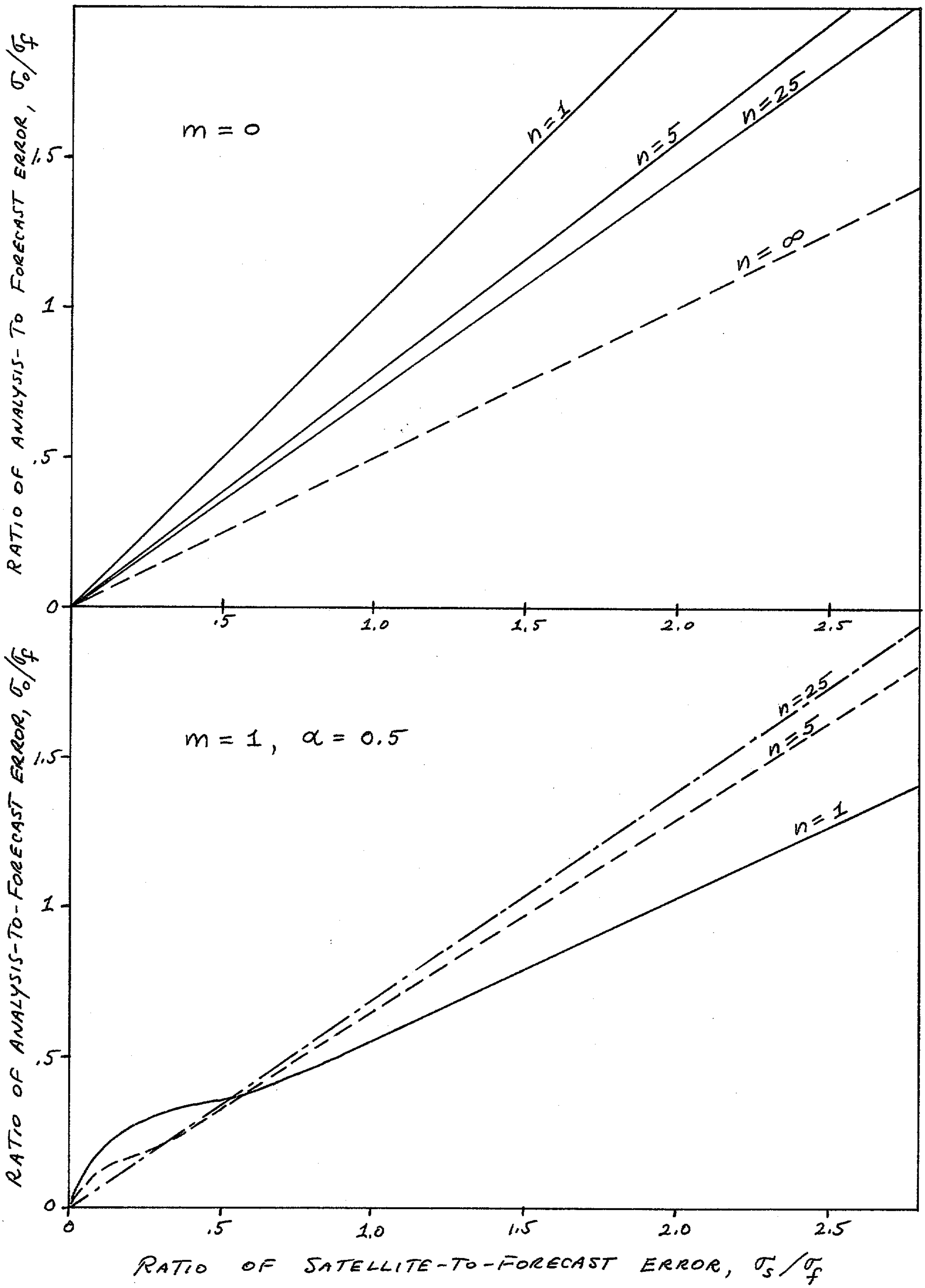

Fig. 13. Ratio of analysis error to forecast error, σ_o/σ_f, as a function of β for equal weighting of all observations, $\rho_{ss} = 0.5$. (a) σ_o/σ_f for n = 1, 5, 25, and infinity when m = 0. (b) σ_o/σ_f for n = 1, 5, and 25 when m = 1, $\alpha = 0.5$.